On the near-equilibrium and near-frozen regions in an expansion wave in a relaxing gas

By J. G. JONES
Royal Aircraft Establishment, Bedford

(Received 14 August 1963)

A weak expansion wave propagating in a relaxing gas is discussed with particular reference to the 'near-equilibrium' and 'near-frozen' regions. The concept of bulk viscosity is used in conjunction with Burgers's equation in the near-equilibrium region. The asymptotic equilibrium simple wave is modified by diffusive regions in the neighbourhood of the first and last rays. It is shown that in the case of a weak expansion wave, Chu's asymptotic solution of the acoustic equation describes the wave-form for a finite time interval before convection effects become noticeable. In the near-frozen region a characteristic perturbation method is used to describe the flow near the wave-front.

1. Introduction

This paper is concerned with the near-equilibrium and near-frozen regions of an expansion wave generated in a relaxing gas by impulsively retracting a piston.

Previous work on the subject (together with the analogous problem of two-dimensional steady supersonic flow past a corner) includes analytical treatment by Chu (1958), Clarke (1960), and Moore & Gibson (1959), using the linearized equations of motion (acoustic approximation), neglecting the effects of convection. Numerical solutions of the full equations have been published by Wood & Parker (1958) and Appleton (1960).

Here, analytical results are obtained which include the effect of convection, the near-equilibrium region (asymptotic solution for large time t) being discussed using the concept of bulk-viscosity, and the near-frozen region (small values of t) using a characteristic perturbation method.

If the terms in the equations of motion are linearized, thus neglecting the effects of convection, there results the acoustic approximation:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial t^2}-a_\infty^2\frac{\partial^2 v}{\partial x^2}\right)+\frac{1}{\tau}\left(\frac{\partial^2 v}{\partial t^2}-a_0^2\frac{\partial^2 v}{\partial x^2}\right)=0, \tag{1}$$

where (x,t) are respectively space and time co-ordinates, v is fluid velocity, τ is relaxation time, and the constants a_∞, a_0 are respectively the 'frozen' and 'equilibrium' speeds of sound. Equation (1) belongs to a class of equations, discussed by Whitham (1959), in which the propagation speed a_∞ defined by the highest-order derivatives (i.e. the 'characteristic' speed) is greater than the speed a_0 at which the main disturbance eventually travels. In the case of an expansion wave, supposed generated at $t=0$, the parameter τ/t affects the nature of equation (1). For small τ/t, i.e. large enough values of t, the lower-order terms dominate. The resulting wave travels at the equilibrium speed of sound (figure 1), the higher-order terms in equation (1) producing a diffusive effect. The asymptotic shape of the wave, of 'error-function' wave-form, was derived by Chu (1958).

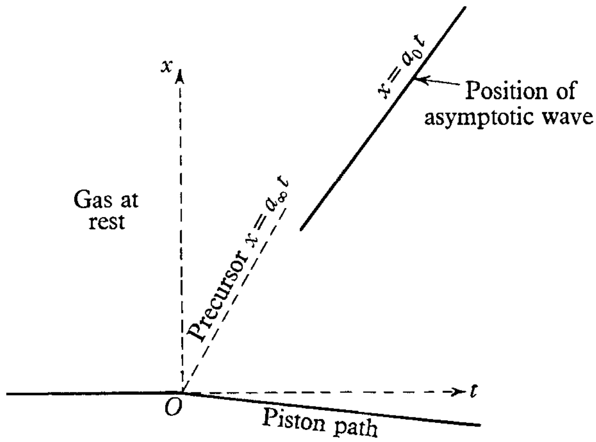

FIGURE 1. Acoustic approximation.

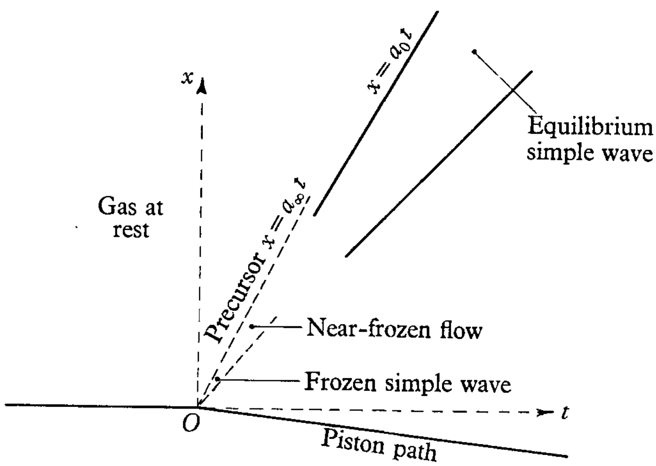

FIGURE 2. Expansion wave with relaxation.

For large values of τ/t, i.e. small enough t, the high-order terms in equation (1) dominate and the flow is of 'frozen' character. The wave-front, or precursor, travels at the frozen speed of sound a_∞ (figure 1). The low-order terms produce an exponential damping effect (Whitham 1959).

When convection terms are retained in the equations of motion, a rather different picture, described by Wood & Parker (1958), emerges. Figure 2 illustrates the flow-field schematically. The asymptotic solution for large values of t is the well-known (equilibrium flow) centred simple wave, or Prandtl–Meyer expansion, the wave-front travelling at the equilibrium speed of sound. Thus convection eventually dominates the flow-field. The discussion of the near-equilibrium region will therefore be concerned with finding the way in which this equilibrium solution is approached. It will be shown that, in the case of weak expansion waves generated by retracting the piston with small velocity, Chu's (1958) asymptotic solution of the linearized equations in fact is valid for a finite time interval during which t is large enough for the asymptotic approximation to be valid, and at the same time small enough for the effects of convection to be negligible. This solution will be used in establishing the correct asymptotic wave-form by providing a time interval in which the linearized solution can be matched with the convective solution, thus providing 'boundary conditions' for the latter. The asymptotic solution is found as a solution of Burgers's equation, a method introduced by Lighthill (1956) for the treatment of shock-wave structure.

For small values of t, the 'frozen-flow' solution takes the form of a centred simple wave with the wave-front travelling at the frozen speed of sound (figure 2). This solution is used to find the 'near-frozen' solution, which describes the flow field for slightly larger values of t, by providing an approximate value for the energy transfer from the 'lagging' to the 'active' mode of the gas. A characteristic perturbation method is used which gives, in particular, the equation of the out-going characteristics in the near-frozen region explicitly. In order to obtain simple analytical results attention is restricted to the flow near the wave-front. This problem has previously been considered by Wood & Parker (1958) (who use Lagrangian equations), and also by Napolitano (1960) who gives a mathematical method for treating the near-equilibrium and near-frozen regions of an expansion wave in the case of an ideal dissociating gas, obtaining series expansions of the full equations which apply in the two regions. The method has, however, only been carried through to obtain an explicit solution in the case of the near-frozen flow. An expansion technique similar to Napolitano's has also been given for near-frozen flow of an ideal dissociating gas by Stulov (1962).

2. The relaxing gas

Typical instances of relaxing gases are those with 'lag' associated with the vibrational modes of molecules, and with dissociation processes. We confine our attention here to the case of vibrational lag, although the solutions obtained have analogous counterparts in the case of dissociation. The equations describing a gas with a lagging mode have been described by Lighthill (1956) whose terminology we follow closely here.

When the gas is in equilibrium, the portion c_l of the specific heat at constant volume, c_v, which is associated with the lagging mode can be written

$$c_l/c_v = (\gamma - 1)\, F_l(T), \tag{2}$$

where T is temperature, and γ is the ratio of specific heats at constant pressure and constant volume. Then, in the case of a vibrational mode,

$$F_l(T) = (T_l/T)^2\, e^{-T_l/T}, \tag{3}$$

where T_l is 'excitation temperature'. When the gas goes through a succession of equilibrium states a small change in internal energy e is given by

$$de = \{\tfrac{5}{2} + F_l(T)\} d(p/\rho). \tag{4}$$

The energy in the vibrational mode, e_l per unit mass, is thus related to translation energy $\frac{3}{2}p/\rho$ by the equation

$$e_l = E_l(p/\rho), \tag{5}$$

where

$$E_l'(p/\rho) = F_l(T). \tag{6}$$

Except in the case of extremely slow changes in p/ρ the energy in the lagging mode is not given by equation (5) but has to be found from the 'rate equation'

$$\frac{de_l}{dt} = \frac{E_l(p/\rho) - e_l}{\tau}, \tag{7}$$

where τ is the relaxation time. The validity of equation (7) is discussed by Lighthill (1956).

In general τ is a function of temperature and pressure, but in the case of a weak expansion wave it has been assumed that the variations in p/ρ are sufficiently small for τ to be regarded as constant.

3. Near equilibrium flow

3.1. *Equations of motion*

The equations of motion, in the case of near-equilibrium flow (neglecting effects of viscosity and heat conduction) have been presented by Lighthill (1956) in the form

$$D\rho/Dt + \rho \operatorname{div} \mathbf{v} = 0, \tag{8}$$

$$\frac{Dv_i}{Dt} + \frac{1}{\rho}\frac{\partial p_{ij}}{\partial x_j} = 0, \tag{9}$$

$$\frac{De}{Dt} = -\frac{1}{\rho} p_{ij} e_{ij}, \tag{10}$$

$$e = \frac{1}{\gamma - 1}\frac{p}{\rho}, \tag{11}$$

where e_{ij} is the 'rate of strain' tensor

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{12}$$

and p_{ij} is the stress tensor

$$p_{ij} = (p - \mu_v \Delta)\,\delta_{ij}. \tag{13}$$

Here, Δ is the dilatation rate

$$\Delta = e_{KK} \tag{14}$$

(using summation suffix convention) and, when $\Delta > 0$, for example, the thermodynamic pressure $p = (\gamma - 1)\rho e$ remains, owing to the effect of 'lag', slightly greater than $\frac{1}{3}p_{KK}$ by an amount $\mu_v \Delta$. μ_v is the 'bulk-viscosity'.

By comparing the structure of a weak shock wave in a relaxing gas with that of a weak shock wave in a viscous gas Lighthill (1956) showed that the appropriate value for the bulk-viscosity is†

$$\mu_v = \rho\tau(a_\infty^2 - a_0^2). \tag{15}$$

Here the amplitude of the wave is small, and the values of a_∞, a_0 are those corresponding to conditions in the undisturbed gas. The same result (equation (15)) has been deduced by Landau & Lifshitz (1959) by considering a periodic compression and expansion wave in a relaxing gas using linearized equations. In the case of near-equilibrium flow (which occurs when the relaxation time is small relative to the period of the motion) they show that the form of the stress tensor is identical with that arising in the theory of viscous flow, the viscosity being replaced by the bulk-viscosity given by equation (15).

In the case of waves of small amplitude (corresponding in our problem to small piston velocity) a simplification can be introduced (Lighthill 1956) in the equations of motion by neglecting terms involving the squares of the (small) velocity amplitude and effective frequency (the latter being proportional to mean velocity gradient). For a weak expansion wave it will be verified in §§ 3.4 and 3.5 that a sufficient condition for this approximation to be applicable throughout the flow field, except in a 'boundary layer' in the neighbourhood of the piston, is that the piston velocity v_1 satisfy the inequality

$$v_1 \ll (a_\infty^2 - a_0^2)^{\frac{1}{2}}. \tag{16}$$

Making the above simplification Lighthill (1956) derived Burgers's equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial X} = \tfrac{1}{2}\delta\frac{\partial^2 u}{\partial X^2} \tag{17}$$

in conjunction with the equation

$$a/(\gamma-1) - \tfrac{1}{2}v = \text{const.} = a_0/(\gamma-1), \tag{18}$$

where

$$u = a + v - a_0, \tag{19}$$

$$X = x - a_0 t, \tag{20}$$

and the fluid velocity and local equilibrium speed of sound are now denoted by v and a respectively, a_0 being the speed of sound in the undisturbed medium. u is then the 'excess wavelet velocity' and X is a co-ordinate in a frame of reference moving in the same direction as the wave at speed a_0. δ is the 'diffusivity', related to bulk viscosity by the equation

$$\delta = \mu_v/\rho = \tau(a_\infty^2 - a_0^2). \tag{21}$$

Equations (18) and (19) can be used to express v and a as linear functions of u. All results concerning wave propagation can thus be completely defined in terms of the variable u.

Burgers's equation (17) is the simplest equation including both non-linear convective terms and (linear) diffusive terms. In the rest of this section it is

† Dr D. A. Spence has pointed out to the author that equation (15) can be derived directly from the rate equation (7) without the need to consider any particular flow process such as a shock or sound wave.

used to discuss the modification, in the near-equilibrium region, of an equilibrium-flow centred simple wave. In the Appendix it is used to discuss the propagation of a non-centred expansion wave in a region where diffusive effects can everywhere be considered as a small perturbation of convective effects.

3.2. *Alternative forms of boundary conditions*

The expansion wave (figure 2) can be generated by retracting a piston impulsively with constant velocity. The boundary conditions are then understood be given in the form of the prescribed velocity on the piston path. However, there are other ways of prescribing 'boundary conditions', all of which are exactly equivalent as far as the flow field in the neighbourhood of the wave is concerned. For example, the velocity distribution at a fixed point (the initial position of the piston) can be prescribed as a function of time in the form of a step-function. If we consider the linearized solution of the problem this formulation is taken as an approximation to that with prescribed velocity on the piston path. Alternatively, the problem can be considered as an 'initial value' problem; the velocity distribution at $t = 0$ (the moment when piston retraction starts) can then be prescribed as a function of position in the form of a step-function. This approach proves to be more useful when we consider the near-equilibrium asymptotic solution.

Not only are the flow fields identical in the neighbourhood of the wave in the above alternative cases, but asymptotically, for large values of t, the corresponding flow fields become identical for all values of x. Consequently, in what follows we will refer to whichever formulation proves most convenient.

3.3. *Asymptotic equilibrium expansion wave*

Asymptotically, for large values of t, the flow becomes of equilibrium character, and the equations of motion simplify to the equations of thermodynamically reversible flow. Suppose that the equilibrium flow equations give an adequate description of the flow for $t \geqslant t_1$. Then there is an intermediate time interval $0 \leqslant t \leqslant t_1$ separating the equilibrium region from the given initial conditions, supposed prescribed at $t = 0$. This intermediate region takes the form of a 'boundary layer' within which higher-order equations must be used. The main problem in finding asymptotic solutions is the determination of suitable boundary conditions for the asymptotic equations within the region in which they apply. In the present case it will be shown by a physical argument that the 'boundary layer' has a negligible effect on the asymptotic equilibrium solution.

Suppose that the 'initial conditions' are prescribed as a step function in the form

$$\text{at} \quad t = 0, \quad v = \begin{cases} v_1(< 0), & \text{for} \quad x < 0, \\ 0, & \text{for} \quad x > 0. \end{cases} \tag{22}$$

The equilibrium equations hold for $t > t_1$ (figure 3). At $t = t_1$ we can find values x_1, x_2 of x (figure 3) such that

$$\text{at} \quad t = t_1, \quad v = \begin{cases} v_1, & \text{for} \quad x < x_2, \\ 0, & \text{for} \quad x > x_1 \end{cases} \tag{23}$$

(a sufficient condition for this is that $x_1 > a_\infty t_1$, $x_2 < -a_\infty t_1$, where a_∞ is the frozen speed of sound in the undisturbed gas).

In seeking an asymptotic solution we look at the flow field on such a large scale that the flow conditions at t_1 (equation (23)) can be regarded as a small perturbation of the given initial conditions (equation (22)). The dominant feature of the conditions at $t = t_1$, as far as the asymptotic solution is concerned, is that outside an interval $x_1 \geqslant x \geqslant x_2$ the initial conditions have not been perturbed, and for large enough t the interval (x_1, x_2) is negligibly small. Moreover, the distinction between $t = 0$ and $t = t_1$ is negligible.

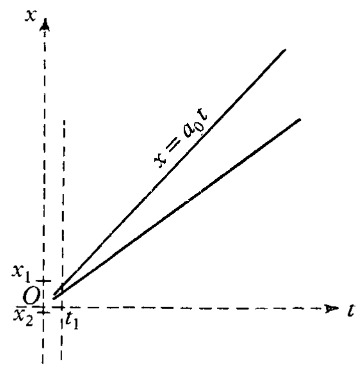

FIGURE 3. Asymptotic equilibrium expansion wave.

The above argument shows that the asymptotic solution for large t can be obtained by using the homentropic flow equations in conjunction with initial conditions given by equation (22) as an adequate approximation to the intermediate conditions which exist at $t = t_1$, described in part by equation (23). That is, the 'boundary layer' has negligible effect. Wood & Parker (1958), in deducing the form of the asymptotic wave, use this result implicitly, without discussion.

The asymptotic solution for large t obtained in this way is simply the equilibrium-flow centred simple wave, or Prandtl–Meyer expansion, with wave-front travelling at the equilibrium speed of sound.

In the above argument it has been assumed that 'initial conditions' have been prescribed in the form of a step function at $t = 0$. A similar argument shows that the result is also true in the case of boundary conditions prescribed on the piston path (or at $x = 0$). In this case the equilibrium flow equations hold for $x \geqslant x_1$ and the 'boundary layer' is the region between the piston and the station $x = x_1$.

3.4. *The 'boundary-layer' problem for the near-equilibrium region*

Just as there exists a station x_1 such that the equilibrium flow equations describe the flow field adequately for $x \geqslant x_1$ there exists a station x_1' where $x_1' < x_1$ such that the near-equilibrium flow equations describe the flow field adequately for $x \geqslant x_1'$. Within the 'boundary layer' $0 \leqslant x \leqslant x_1'$ (where $x = 0$ is the approximate position of the piston), higher-order equations must be used. It will now be shown that, in the case of a weak expansion wave, Chu's (1958) solution of the linearized equations can be used to relate the near-equilibrium flow asymptotic solution to the prescribed boundary conditions.

Chu (1958) considered the case of an expansion wave, with boundary conditions prescribed at the approximate piston position $x = 0$, using linearized equations equivalent to equation (1). He was in fact concerned with a gas with chemical reactions, but his results are equally applicable to the case of a lagging internal mode. He deduced the asymptotic solution

$$v = \tfrac{1}{2}v_1\left[1-\operatorname{erf}\left(\frac{x/a_0-t}{\{2(a_\infty^2/a_0^2-1)\,\tau t\}^{\frac{1}{2}}}\right)\right], \tag{24}$$

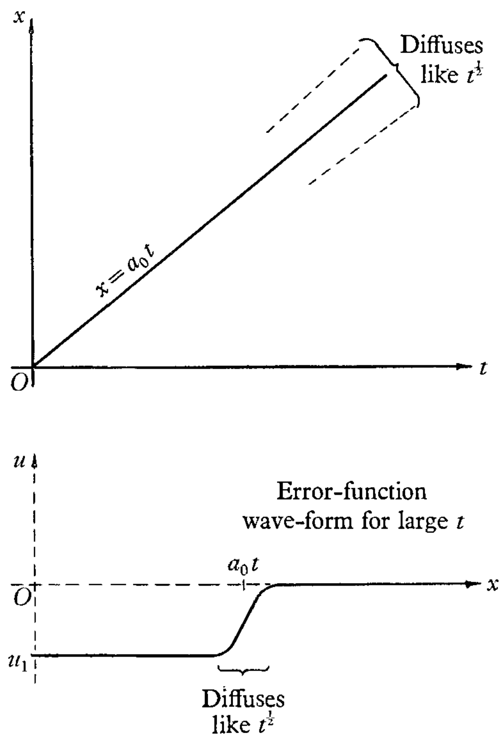

FIGURE 4. Chu's asymptotic solution of the linearized equations.

which is an error-function wave-form travelling at speed a_0 with the effect of diffusion being to spread the initial discontinuity over a length increasing like $\{\delta x/a_0\}^{\frac{1}{2}}$, where δ is 'diffusivity', given by equation (21). This asymptotic wave is illustrated in figure 4. An analogous asymptotic form was derived by Clarke (1960) in the case of two-dimensional steady flow of a dissociating gas past a corner.

Although the effect of convection has been neglected in the derivation of equation (24), in the case of a weak expansion wave it will now be shown that it gives a valid description of the wave-form in part of the flow field. On the one hand t must be large enough to ensure the near-equilibrium character of the flow field. The condition for this is that

$$\tau/t \ll 1. \tag{25}$$

On the other hand, at an instant t when the wave-form is given by equation (24), t must be small enough for the diffusive effect, spread like $t^{\frac{1}{2}}$, to dominate the convective wave-spread, or 'fanning', which gives a wave-form of length proportional to t. As the length of the diffusion-dominated wave-form in equation (24) is proportional to $(\delta t)^{\frac{1}{2}}$ and the convective effect gives rise to a wave of length $v_1 t$ this latter condition can be written in the form

$$v_1 t \ll (\delta t)^{\frac{1}{2}}$$

or

$$v_1 \ll \{(\tau/t)\,(a_\infty^2 - a_0^2)\}^{\frac{1}{2}}. \tag{26}$$

Combining inequalities (25) and (26) it follows that equation (24) will give a valid description of the wave-form in a time interval

$$\tau \ll t \ll \tau(a_\infty^2 - a_0^2)/v_1^2. \tag{27}$$

Such an interval will only exist if the piston velocity is subject to the restriction

$$v_1 \ll (a_\infty^2 - a_0^2)^{\frac{1}{2}}. \tag{28}$$

The inequality (27) can be put in the alternative form

$$a_0 \tau \ll x \ll a_0 \tau(a_\infty^2 - a_0^2)/v_1^2. \tag{29}$$

In the case of a weak expansion wave the station x_1' can be chosen to satisfy inequality (29), the linearized solution of the problem thus holding throughout the 'boundary layer' $0 \leqslant x \leqslant x_1'$. At x_1' the linearized solution takes the form of equation (24). It remains to find a convective solution for the 'outer' region $x_1' \leqslant x$ which matches the linearized solution, i.e. takes the form of equation (24), at x_1'. It will be shown in the following section that such a solution can be found as a solution of Burgers's equation satisfying initial conditions at $t = 0$ in the form of a step function (equation (22)). Of course, this will provide a mathematical solution for the complete region $0 \leqslant x$ but in the interval $0 \leqslant x \leqslant x_1'$ this solution is not physically meaningful in the present context and must be replaced by the solution of the linearized equations (Chu 1958; Clarke 1960) which take full account of the relaxation process. In the following section we use the solution of Burgers's equation derived thus to discuss the way in which the asymptotic equilibrium simple wave-form is approached.

3.5. *Near-equilibrium flow for a weak expansion wave*

As explained in the previous section, the asymptotic solution for large values of t, in the case of a weak expansion wave, is obtained from the solution of Burgers's equation (17), with initial conditions given by equation (22). The required solution has been given by Lighthill (1956):

$$u(X,t) = \frac{u_1}{1+\exp\left[\dfrac{u_1(X-\frac{1}{2}u_1 t)}{\delta}\right]\displaystyle\int_{-X}^{\infty}\exp\left(-\frac{y^2}{2\delta t}\right)dy\Big/\int_{X-u_1 t}^{\infty}\exp\left(-\frac{y^2}{2\delta t}\right)dy}, \tag{30}$$

where

$$u(X,0) = \tfrac{1}{2}(\gamma+1)\,v_1 = \begin{Bmatrix} u_1(X<0), \\ 0(X>0), \end{Bmatrix}$$

and the other variables involved have been defined in §3.1. Lighthill was primarily concerned with the compression case ($u_1 > 0$) leading to shock wave formation. Here we are concerned only with the case $u_1 < 0$.

We first verify that, in the case of a weak expansion wave, equation (30) takes the form of equation (24) in the time interval given by inequality (27), thus matching the present solution with Chu's solution of the linearized equations as described in the previous section.

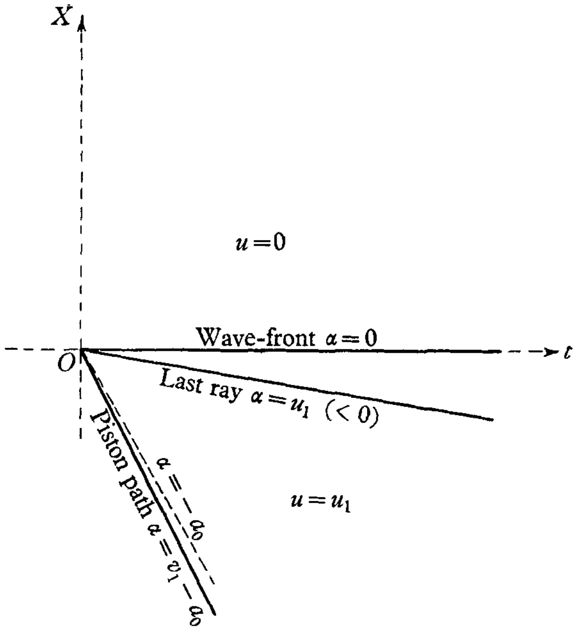

FIGURE 5. Centred simple wave.

In the time interval defined by equation (27) we have the following inequality throughout the wave:

$$X = O(\delta t)^{\frac{1}{2}} \gg u_1 t$$

(except for a negligible region near the wave-centre $X = 0$) and thus equation (30) becomes approximately

$$u = \frac{u_1\{1-\operatorname{erf}[X/(2\delta t)^{\frac{1}{2}}]\}}{\{1-\operatorname{erf}[X/(2\delta t)^{\frac{1}{2}}]\}+\exp(u_1X/\delta)\{1+\operatorname{erf}[X/(2\delta t)^{\frac{1}{2}}]\}}. \tag{31}$$

Also, since $X = O(\delta t)^{\frac{1}{2}}$, we have

$$u_1 X/\delta = O(u_1 t/\delta^{\frac{1}{2}}t^{\frac{1}{2}}) \ll 1,$$

thus

$$\exp(u_1 X/\delta) \doteqdot 1,$$

and hence equation (24) results.

For the rest of this section we consider the asymptotic form of equation (30) as $t \to \infty$. In the first place it is convenient to consider conditions on the ray $X = \alpha t$. Then, writing

$$u(X,t) \equiv \bar{u}(\alpha, t), \tag{32}$$

equation (30) becomes

$$\bar{u}(\alpha,t) = \frac{u_1}{1+\exp\left\{\dfrac{u_1(\alpha-\frac{1}{2}u_1)t}{\delta}\right\}\displaystyle\int_{-\alpha(t/2\delta)^{\frac{1}{2}}}^{\infty} e^{-y^2}\,dy \Bigg/ \int_{(\alpha-u_1)(t/2\delta)^{\frac{1}{2}}}^{\infty} e^{-y^2}\,dy}. \tag{33}$$

Using the asymptotic expansion

$$\int_Y^{\infty} e^{-y^2}\,dy = \frac{e^{-Y^2}}{2Y}\left(1-\frac{1}{2Y^2}+\ldots\right), \tag{34}$$

the following asymptotic forms can be found.

To a first approximation (large t) equation (33) reduces to the equilibrium simple wave produced in an ideal fluid by the initial conditions (equation (22)), viz.

$$\bar{u}(\alpha,t) \doteqdot \begin{cases} 0, & \text{for} \quad \alpha > 0, \\ \alpha, & \text{for} \quad 0 > \alpha > u_1, \\ u_1, & \text{for} \quad u_1 > \alpha, \end{cases} \tag{35}$$

confirming the result deduced from physical arguments in § 3.3. The simple wave-flow field in the above variables is illustrated in figure 5.

The next approximation gives

$$\bar{u}(\alpha,t) \doteqdot \begin{cases} \dfrac{u_1}{\alpha-u_1}\left(\dfrac{\delta}{2\pi t}\right)^{\frac{1}{2}}\exp\left(-\dfrac{\alpha^2 t}{2\delta}\right), & \text{for} \quad \alpha > 0, \\ \alpha\left\{1+\dfrac{\delta}{t}\dfrac{2\alpha-u_1}{\alpha^2(\alpha-u_1)}\right\}, & \text{for} \quad 0 > \alpha > u_1, \\ u_1\left\{1+\dfrac{1}{\alpha}\left(\dfrac{\delta}{2\pi t}\right)^{\frac{1}{2}}\exp\left(-\dfrac{(u_1-\alpha)^2 t}{2\delta}\right)\right\}, & \text{for} \quad u_1 > \alpha. \end{cases} \tag{36}$$

This approximation does not help us to find the shape of the wave-form for given large t, as it is evidently non-uniformly valid near $\alpha = 0$ and $\alpha = u_1$ where the additional terms become infinite. However, it is useful in that it shows that the effect of diffusivity on the solution falls off more slowly (like $1/t$) inside the wave ($0 > \alpha > u_1$) than outside it, where the decay is exponential.

The behaviour of this approximation near the end rays, $\alpha = 0$ and $\alpha = u_1$, suggests that we examine in more detail the flow near these rays. In order to investigate flow conditions in the neighbourhood of the leading ray, $\alpha = 0$, we consider the path in the (x,t)-plane such that

$$\alpha(t/2\delta)^{\frac{1}{2}} = \text{const.} = A, \quad \text{say}. \tag{37}$$

Since $X = \alpha t$ we have

$$X = A(2\delta t)^{\frac{1}{2}}. \tag{38}$$

Now X is the distance of a point ahead of the wave-front, so that as $t \to \infty$ the distance of the path A = const. from the wave-front increases like $t^{\frac{1}{2}}$. At the same time, however, $\alpha \to 0$. Writing

$$\bar{\bar{u}}(A,t) \equiv \bar{u}(\alpha,t), \tag{39}$$

equation (33) reduces to the asymptotic form (for A fixed, as $t \to \infty$)

$$u(A,t) \doteqdot -\left(\frac{2\delta}{t}\right)^{\frac{1}{2}}\left[\frac{e^{-A^2}}{\pi^{-\frac{1}{2}}\{1-\operatorname{erf}(-A)\}}\right]. \tag{40}$$

The term in square brackets gives the shape of the wave-front as a function of A. From equations (38) and (40), then, it follows that u decays like $t^{-\frac{1}{2}}$ near $\alpha = 0$ and that the neighbouring region within which the effect of diffusivity is significant is spread over a length increasing like $t^{\frac{1}{2}}$. A similar result holds near the end ray $\alpha = u_1$. By comparison with the results expressed in equation (36),

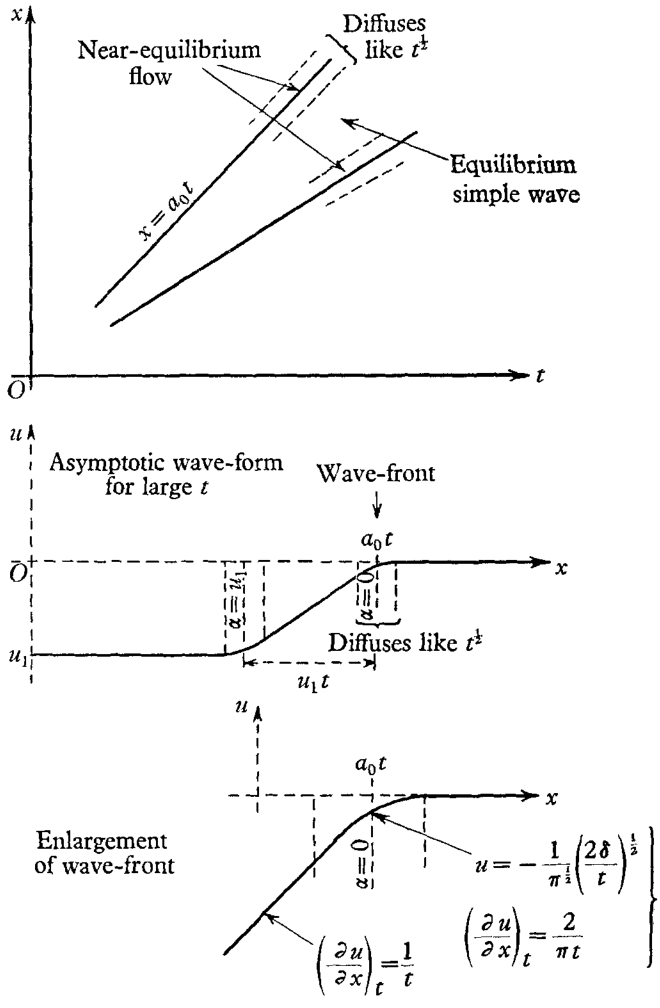

FIGURE 6. Asymptotic solution for large t.

which indicates exponential decay outside the wave and decay like t^{-1} inside the wave, it can now be seen that the near-equilibrium region, for large values of t, is concentrated in regions near the end rays (figure 6). The $t^{-\frac{1}{2}}$ decay in these regions explains the singularities in the coefficients of t^{-1} at $\alpha = 0$ and $\alpha = u_1$ observed in the asymptotic expansion given in equation (36). Since the length of the wave (from first to last ray) increases like t as $t \to \infty$ (figure 6) the length of the diffusive regions tends relatively to zero.

In order to find the shape of the wave-form near the wave-front for large t

it is convenient to approximate to equation (40) by considering A small. The result can be written in the form

$$u \doteqdot -\frac{1}{\pi^{\frac{1}{2}}}\left(\frac{2\delta}{t}\right)^{\frac{1}{2}} + \frac{2\alpha}{\pi} \tag{41}$$

(where $\alpha \to 0$ like $t^{-\frac{1}{2}}$ as $t \to \infty$). In particular, putting $\alpha = 0$ in equation (41), the decay at the leading ray is obtained. The slope of the wave-form near the wave-front (A small) can also be deduced, from equation (41), as

$$\left(\frac{\partial u}{\partial x}\right)_t = \left(\frac{\partial u}{\partial \alpha}\right)_t \left(\frac{\partial \alpha}{\partial x}\right)_t = \frac{2}{\pi t}. \tag{42}$$

This result may be contrasted with the slope outside the diffusive region, where the simple-wave approximation, equation (35), is valid:

$$(\partial u/\partial x)_t = 1/t. \tag{43}$$

The information contained in equations (35) and (40) has been combined to give the shape of the asymptotic wave-form in figure 6.

3.6. *Near-equilibrium flow for a strong expansion wave*

In the case of a weak expansion wave it has been shown in the previous section how the asymptotic equilibrium-flow simple wave is modified in the near-equilibrium region. Here we use these results tentatively to suggest the probable form of the near-equilibrium region in the case of a strong expansion wave.

In the first place it seems likely that, although the quantitative results obtained for a weak expansion wave no longer apply directly, the general result that the equilibrium simple wave is modified in the near-equilibrium region by diffusive bands in the neighbourhood of the first and last rays still holds. If we make this assumption quantitative results can again be obtained. For, in the neighbourhood of the leading ray for instance, the velocity amplitude of the wave is small and Burgers's equation is again valid. Moreover, we seek a solution of Burgers's equation which joins the flow conditions in the gas at rest upstream of the wave to the flow conditions in the equilibrium simple wave behind the leading diffusive band. But the latter conditions are identical with those in a weak simple wave, therefore the solution required is exactly that already obtained for a weak expansion wave, the wave-form being given by equation (40) (and illustrated in figure 6), the value of δ being taken as that appropriate to flow conditions in the undisturbed gas.

An analogous argument applies in the case of the diffusive band centred on the last ray of the equilibrium simple wave. Here, flow conditions are small perturbations of (equilibrium) conditions in the gas behind the wave. So again Burgers's equation can be used to join the flow conditions in the equilibrium simple wave just upstream of the last ray to the flow conditions in the gas downstream of the wave. Moreover, the weak expansion wave again provides the necessary diffusive wave-form, the value of δ now, however, referring to conditions in the gas behind the wave.

To summarize, it appears that in the case of a strong expansion wave the results illustrated in figure 6 can be simply adjusted to give the flow field in the near-equilibrium region. The equilibrium flow solution is modified by diffusive bands exactly as in figure 6, except that in the leading band the value of δ is taken as that appropriate to conditions in the undisturbed gas, and in the trailing band the value of δ is taken as that appropriate to the (equilibrium) conditions behind the wave.

4. Near frozen flow

The known frozen flow solution (very small values of t) is now used to find the flow field in the near-frozen region (slightly larger values of t). In order to obtain simple analytical results attention is restricted to the flow near the wave-front.

4.1. *Equations of motion*

For very small values of t the equations of motion reduce to the equations of thermodynamically reversible flow, the ratio of specific heats of the gas being that (γ') applicable to the 'active' mode only. This is the 'frozen' flow region. The solution of the equations in the case of an impulsively retracted piston is a centred simple wave (Wood & Parker 1958; Napolitano 1960).

It is the flow field for slightly larger values of t, the 'near-frozen' region, that concerns us here. The method employed is to use the frozen flow simple wave to provide a first approximation for the energy transfer from the lagging to the active mode. The equations of motion are those for the active mode with energy transfer from the lagging mode regarded as external heat addition. These equations have been derived in convenient characteristic form by Kantrowitz (1958) and will be used in what follows with some differences of notation (designed to avoid using the concept of 'temperature').

The frozen speed of sound (a function of x and t) is given by

$$a_\infty = \{\gamma' p/\rho\}^{\frac{1}{2}}. \tag{44}$$

Writing

$$n' = 2/(\gamma'-1), \tag{45}$$

we introduce 'Riemann invariants'

$$r = \tfrac{1}{2}(n'a_\infty + v), \quad s = \tfrac{1}{2}(n'a_\infty - v). \tag{46}$$

The equations of motion can then be written in the form (Kantrowitz 1958)

$$\delta^+ r = \frac{a_\infty}{2\gamma' p/\rho}\left\{\delta^+ q + (\gamma'-1)\left(\frac{Dq}{Dt}\right)\delta t\right\}, \tag{47}$$

$$\delta^- s = \frac{a_\infty}{2\gamma' p/\rho}\left\{\delta^- q + (\gamma'-1)\left(\frac{Dq}{Dt}\right)\delta t\right\}, \tag{48}$$

where q is the energy transferred to the active mode,

$$\frac{\delta^+}{\delta t} = \frac{\partial}{\partial t} + (v + a_\infty)\frac{\partial}{\partial x} \tag{49}$$

and

$$\frac{\delta^-}{\delta t} = \frac{\partial}{\partial t} + (v - a_\infty)\frac{\partial}{\partial x} \tag{50}$$

are derivatives along characteristics, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x} \tag{51}$$

is the substantial derivative.

4.2. *Approximate evaluation of energy transfer*

The energy transfer dq to the active mode is given by

$$dq = -de_l \tag{52}$$

where
$$\frac{de_l}{dt} = \frac{E_l(p/\rho) - e_l}{\tau}. \tag{7}$$

In the frozen flow region t/τ is so small that the right-hand side of equation (7) is negligible:

$$\{de_l/dt\}_f = 0, \tag{53}$$

where suffix f refers to frozen flow. Thus

$$\{e_l\}_f = \text{const.} = E_l(\bar{p}/\bar{\rho}), \tag{54}$$

where a bar denotes conditions in undisturbed gas at rest.

The approximate energy transfer in the near-frozen (nf) region can now be obtained from equation (7) in the form

$$\begin{aligned}\left\{\frac{de_l}{dt}\right\}_{nf} &\doteqdot \left\{\frac{E_l(p/\rho) - e_l}{\tau}\right\}_f \\ &= \tau^{-1}\left[\{E_l(p/\rho)\}_f - E_l(\bar{p}/\bar{\rho})\right],\end{aligned} \tag{55}$$

using equation (54).

4.3. *Frozen and near-frozen flow fields*

In the undisturbed gas at rest (conditions denoted by a bar) we have

$$r = \bar{r} = \tfrac{1}{2}n'\bar{a}_\infty, \quad s = \bar{s} = \tfrac{1}{2}n'\bar{a}_\infty. \tag{56}$$

The frozen simple wave solution (denoted by suffix zero) can be written in the form

$$r = r_0, \quad s = s_0 = \tfrac{1}{2}n'\bar{a}_\infty. \tag{57}$$

s_0 is thus constant throughout the flow field and r_0 is constant along out-going (straight) characteristics (figure 7). In order to obtain simple analytical results we restrict attention to flow near the wave-front and regard small terms, $O(1 - 2r_0/n'\bar{a}_\infty)^2$, as negligible. The frozen wave-front travels at speed

$$(\partial x/\partial t)_{\bar{r}_0} = \bar{a}_\infty. \tag{58}$$

The outgoing characteristics r_0 travel at speed

$$(\partial x/\partial t)_{r_0} = (a_\infty + v)_0 = \bar{a}_\infty - \tfrac{1}{2}\bar{a}_\infty(1+n')(1 - 2r_0/n'\bar{a}_\infty), \tag{59}$$

(from equations (46) and (57)).

Suppose now (figure 7) that the frozen wave-front and the frozen characteristic

r_0 reach position x at times $t_0(x)$ and $t(x, r_0)$ respectively. Then from equations (58) and (59) it follows that

$$t(x, r_0) = \int_0^x \left(\frac{\partial x}{\partial t}\right)_{r_0}^{-1} dx \doteqdot t_0(x) + \left(\frac{1+n'}{2\bar{a}_\infty}\right)\left(1 - \frac{2r_0}{n'\bar{a}_\infty}\right)x, \tag{60}$$

to first order in $(1 - 2r_0/n'\bar{a}_\infty)$. Equation (60) is the approximate equation for the frozen outgoing characteristics.

We also have

$$\{p/\rho\}_f = (\bar{p}/\bar{\rho})(a_\infty/\bar{a}_\infty)^2 \doteqdot (\bar{p}/\bar{\rho})\{1 - (1 - 2r_0/n'\bar{a}_\infty)\}. \tag{61}$$

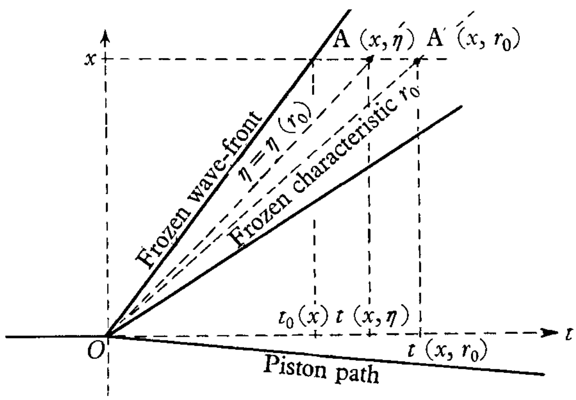

FIGURE 7. Near-frozen centred expansion.

From equations (55) and (61) there results

$$\left\{\frac{de_l}{dt}\right\}_{nf} \doteqdot \frac{-1}{\tau}\left(\frac{\bar{p}}{\bar{\rho}}\right) E_l'\left(\frac{\bar{p}}{\bar{\rho}}\right)\left(1 - \frac{2r_0}{n'\bar{a}_\infty}\right). \tag{62}$$

To first order in $(1 - 2r_0/n'\bar{a}_\infty)$ the particle paths have equations

$$x = \text{const.}, \tag{63}$$

hence integration of equation (62) gives

$$\begin{aligned}\{e_l\}_{nf} &\doteqdot \int_{t_0(x)}^{t(x, r_0)} \left\{\frac{de_l}{dt}\right\}_{nf} dt = \int_{\frac{1}{2}n'\bar{a}_\infty}^{r_0} \left\{\frac{de_l}{dt}\right\}_{nf} \left(\frac{dt}{dr_0}\right)_x dr_0 \\ &= -\frac{1}{\tau}\left\{E_l'\left(\frac{\bar{p}}{\bar{\rho}}\right)\right\}\left(\frac{\bar{p}}{\bar{\rho}}\right)\frac{(1+n')}{4\bar{a}_\infty}\left(1 - \frac{2r_0}{n'\bar{a}_\infty}\right)^2 x\end{aligned} \tag{64}$$

(using equation (60)).

In the near-frozen flow the frozen characteristic r_0 (figure 7) will be perturbed, although its initial ($t = 0$) slope will remain unchanged. Suppose this perturbed outgoing characteristic has equation

$$\eta = \text{const.}, \tag{65}$$

where η and r_0 are related by the equation

$$r_0 = r_0(\eta). \tag{66}$$

Then we can take parametric co-ordinates (x, η) in the near-frozen expansion wave. First, flow conditions at the point $\mathrm{A}(x, \eta)$ (figure 7) will be determined, and then the position of this point in the (x, t)-plane will be found to a first approximation.

Using the approximate form (63) for the particle paths, equation (51) becomes

$$D/Dt \doteqdot (\partial/\partial t)_x. \tag{67}$$

Equation (47) can now be written in the form

$$\frac{\partial r(x,\eta)}{\partial x} = \frac{a_\infty}{2\gamma' p/\rho}\left\{\frac{\partial q(x,\eta)}{\partial x} + (\gamma'-1)\left(\frac{\partial q}{\partial t}\right)_x \frac{\partial t(x,\eta)}{\partial x}\right\}. \tag{68}$$

To first order we have

$$q(x,\eta) \doteqdot q(x, r_0). \tag{69}$$

Equations (52), (62), (64) and (68) thus give, to first order in $(1 - 2r_0/n'\bar{a}_\infty)$,

$$\frac{\partial r(x,\eta)}{\partial x} = \frac{1}{2\tau}\frac{\gamma'-1}{\gamma'} E_l'\left(\frac{\bar{p}}{\bar{\rho}}\right)\left(1 - \frac{2r_0(\eta)}{n'\bar{a}_\infty}\right), \tag{70}$$

$r_0(\eta)$ being defined in equation (66). Integrating equation (70), we obtain

$$r(x,\eta) = r_0(\eta) + \frac{1}{2\tau}\frac{\gamma'-1}{\gamma'} E_l'\left(\frac{\bar{p}}{\bar{\rho}}\right)\left(1 - \frac{2r_0(\eta)}{n'\bar{a}_\infty}\right) x. \tag{71}$$

To the same order of approximation it follows from equation (48) that

$$s(x,\eta) = \bar{s} = \tfrac{1}{2}n'\bar{a}_\infty. \tag{72}$$

Equations (71) and (72) determine all flow variables in the near-frozen flow field at the point $\mathrm{A}(x, \eta)$.

The velocity of the characteristic $\eta = \text{const.}$ at $\mathrm{A}(x, \eta)$ is

$$\begin{aligned}(\partial x/\partial t)_\eta &= a_\infty(x,\eta) + v(x,\eta) \\ &= (\partial x/\partial t)_{r_0} + (1 + 1/n')\{r(x,\eta) - r_0(\eta)\},\end{aligned} \tag{73}$$

where $(\partial x/\partial t)_{r_0}$ is given by equation (59). Substituting from equation (71) and integrating, the time $t(x, \eta)$ at which the characteristic $\eta = \text{const.}$ reaches position x is given by

$$\begin{aligned} t(x,\eta) &= \int_0^x \left(\frac{\partial x}{\partial t}\right)_\eta^{-1} dx \\ &\doteqdot t(x, r_0(\eta)) - \frac{1}{\tau}\frac{1}{4\bar{a}_\infty^2}\left(1 + \frac{1}{n'}\right)\left(\frac{\gamma'-1}{\gamma'}\right) E_l'\left(\frac{\bar{p}}{\bar{\rho}}\right)\left(1 - \frac{2r_0(\eta)}{n'\bar{a}_\infty}\right) x^2 \\ &\doteqdot t_0(x) + \left(\frac{1+n'}{2\bar{a}_\infty}\right)\left(\frac{1 - 2r_0(\eta)}{n'\bar{a}_\infty}\right)\left\{x - \frac{1}{\tau}\frac{1}{2\bar{a}_\infty n'}\left(\frac{\gamma'-1}{\gamma'}\right) E_l'\left(\frac{\bar{p}}{\bar{\rho}}\right) x^2\right\},\end{aligned} \tag{74}$$

(using equation (60)). Equation (74) (the approximate equation of an outgoing characteristic in the near-frozen flow field) locates the point $\mathrm{A}(x, \eta)$ in the (x, t)-plane and thus completes the parametric solution. It can be seen from equation (74) that, apart from the wave front, the near-frozen flow characteristics travel with greater velocity than their counterparts in the frozen simple wave (owing to the transfer of energy from the lagging to the active mode).

5. Conclusions

The near-equilibrium and near-frozen regions of an expansion wave propagating in a relaxing gas have been discussed. For a wave generated by impulsively retracting a piston at time $t = 0$ these regions occur respectively for large and small values of t.

The asymptotic equilibrium flow field for large values of t is a centred simple wave with wave-front travelling at the equilibrium speed of sound. In the case of a weak expansion wave (piston retracted with small velocity) it has been shown by means of a solution of Burgers's equation that in the near-equilibrium region the simple wave flow is modified by diffusive regions in the neighbourhood of the first and last rays. The resulting wave-form is illustrated in figure 6. The diffusive regions are spread over a length of the wave proportional to $t^{\frac{1}{2}}$ and the diffusive effect decays like $t^{-\frac{1}{2}}$. Since the total length of the wave is proportional to t these diffusive bands have negligible effect on the shape of the wave for very large values of t. It has been shown that, whereas the slope $(\partial u/\partial x)_t$ of the wave-form in the simple wave region is t^{-1}, the slope at the leading ray, within the near-equilibrium region, is $2(\pi t)^{-1}$. This near-equilibrium asymptotic solution, dominated by convection, is to be contrasted with the asymptotic wave-form obtained by Chu (1958) using the acoustic approximation (and thus neglecting convection effects) (figure 4). This takes the form of an 'error-function' diffusive region, of length proportional to $t^{\frac{1}{2}}$, centred on the equilibrium Mach line. However, it has been shown that, in the case of a weak expansion wave, this type of wave-form exists for a finite interval of time before convection effects become noticeable.

The asymptotic solution for a weak expansion wave has been used to suggest the probable form of the near equilibrium regions in the case of a strong expansion wave (piston retracted with large velocity). The diffusive regions centred on the first and last rays of the simple wave are in this case described using values of the 'diffusivity' corresponding respectively to conditions in the gas in front of and behind the wave.

In the Appendix, the near-equilibrium flow associated with an initial non-centred expansion wave-form, in which gradients are sufficiently small for the effects of convection to dominate the whole flow field, is discussed. It is shown that in this case the diffusive decay at first is everywhere like t^{-1}. Asymptotically, however, the wave takes the form of a centred simple wave with diffusive decay like $t^{-\frac{1}{2}}$ in the neighbourhood of the first and last rays as before.

In the case of near-frozen flow (figure 7) a characteristic perturbation method has been employed to describe the flow near the wave-front taking the initial frozen simple wave as a first approximation. The solution representing the initial decay of the frozen wave is expressed parametrically in terms of a variable which is constant on the perturbed outgoing characteristics. The equations of the approximate characteristics have been found explicitly. Apart from the wave-front, which is unperturbed, the characteristics propagate with greater velocity than in the frozen region owing to the transfer of energy from the lagging mode to the active mode.

Appendix. Near-equilibrium non-centred expansion wave

In § 3 Burgers's equation was used to find the near-equilibrium asymptotic approximation to a centred expansion wave for large values of t. In this Appendix we consider an initial non-centred expansion wave-form with gradients sufficiently small for the near-equilibrium assumption to hold for all values of t. Burgers's equation is then used to discuss the development of such a waveform with time, the diffusive effects being regarded as small perturbations of the convective effects. In § A 1 it is assumed that the curvature of the initial wave-form is everywhere small. In § A 2 the wave-form for large t is discussed. In § A 3 the effect of a discontinuity in the slope of the initial wave-form is considered briefly.

A 1. *Curvature of initial wave-form everywhere small*

The solution of Burgers's equation (17) for an arbitrary initial ($t = 0$) wave-form is given by (Lighthill 1956)

$$u(X,t) = I_1/I_2, \tag{75}$$

where

$$I_1 = \int_{-\infty}^{\infty} \frac{X-Y}{t} \exp\left(\frac{F}{\delta}\right) dY, \tag{76}$$

$$I_2 = \int_{-\infty}^{\infty} \exp\left(\frac{F}{\delta}\right) dY, \tag{77}$$

$$F(X, Y, t) = \int_{Y}^{\infty} u(Y, 0)\, dY - \frac{(X-Y)^2}{2t}, \tag{78}$$

and the variables involved have already been defined in § 3.1.

We use the fact that in near-equilibrium flow τ is effectively small, and hence, from equation (21), diffusivity δ is small. In this case, as shown by Lighthill (1956), equation (75) can be approximated by means of the 'method of steepest descents'. Lighthill evaluated the first approximation for general flows, and the second approximation in the neighbourhood of shock waves. We here find the second approximation in the case of an expansion wave.

The first approximation to equation (75) is (Lighthill 1956)

$$u(x,t) = (X - Y_M)/t, \tag{79}$$

where Y_M is the value of Y for which F (equation (78)) is greatest for given values of X and t. Then Y_M is given by

$$F'(Y_M) = -u(Y_M, 0) + (X - Y_M)/t = 0. \tag{80}$$

Provided no shock waves appear in the flow field this solution is identical with that of pure convection theory,

$$u = \text{const. on wavelets}, \quad X - ut = \text{const.} \tag{81}$$

This first approximation then describes an equilibrium simple wave, and the next term in the approximation, which is to be found, describes a small perturbation of it. The lines $X - ut = \text{const.}$ are sometimes called the 'sub-characteristics' of the solution.

Lighthill (1956) explains how the presence of shock waves in the flow field is associated with the function F (equation (78)) having several local maxima for given (X, t). The second approximation in the neighbourhood of a shock wave is found by considering contributions to the function $u(X, t)$ (equation (75)) from each of these maxima. In the case of an expansion wave, a single maximum, at $Y = Y_M$ (say), dominates the value of the integrals in equation (75). The method of obtaining higher approximations in this case using 'steepest descents' is described by Erdelyi (1956).

The condition that $Y = Y_M$ gives a maximum of F requires:

$$F''(Y_M) = -u'(Y_M, 0) - 1/t < 0, \tag{82}$$

i.e.

$$1 + tu'(Y_M, 0) > 0. \tag{83}$$

But, as Lighthill shows, simple convection theory gives

$$1 + tu'(Y_M, 0) = u'(Y_M, 0)/u'(X, t), \tag{84}$$

where

$$u'(X, t) = \partial u(X, t)/\partial X. \tag{85}$$

So, as we are considering the small perturbation of a simple wave expansion, equation (83) is evidently satisfied.

In the neighbourhood of $Y = Y_M$ we have the Taylor expansion

$$F(Y) = F(Y_M) + \tfrac{1}{2}F''(Y_M)(Y - Y_M)^2 + F_1(Y), \tag{86}$$

where

$$F_1(Y) = (1/3!)\,F'''(Y_M)(Y - Y_M)^3 + (1/4!)\,F^{(\mathrm{iv})}(Y_M)(Y - Y_M)^4 + \dots. \tag{87}$$

Substituting this expansion into equations (76) and (77) gives (this procedure is discussed by Erdelyi 1956)

$$I_1 = \left\{\frac{2\pi\delta}{-F''(Y_M)}\right\}^{\frac{1}{2}} \exp\left\{\frac{F(Y_M)}{\delta}\right\} \times \left[\frac{X - Y_M}{t} + \left(\frac{X - Y_M}{t}\frac{F^{(\mathrm{iv})}(Y_M)}{4!\delta} - \frac{F'''(Y_M)}{3!\,\delta t}\right)\frac{3\delta^2}{\{F''(Y_M)\}^2} + \dots\right], \tag{88}$$

and

$$I_2 = \left\{\frac{2\pi\delta}{-F''(Y_M)}\right\}^{\frac{1}{2}} \exp\left\{\frac{F(Y_M)}{\delta}\right\}\left[1 + \frac{F^{(\mathrm{iv})}(Y_M)}{4!\delta}\frac{3\delta^2}{\{F''(Y_M)\}^2} + \dots\right]. \tag{89}$$

So, to a second approximation, from equation (75)

$$u(X, t) = \frac{X - Y_M}{t} - \frac{1}{2}\frac{F'''(Y_M)}{t}\frac{\delta}{\{F''(Y_M)\}^2}, \tag{90}$$

or

$$u(X, t) = \frac{X - Y_M}{t} + \frac{1}{2}\frac{u''(Y_M, 0)}{u'(Y_M, 0)}\,\delta\left[\frac{1}{u'(Y_M, 0)\,t}\left\{1 + \frac{1}{u'(Y_M, 0)\,t}\right\}^{-2}\right]. \tag{91}$$

The first term represents the basic simple wave. The second term, representing the effect of diffusivity, relates the value of u at (X, t) to the shape of the initial wave-form at the point $X = Y_M$, i.e. at the point of the initial wave-form that lies on the sub-characteristic through (X, t). Note that the second term in equation (91) vanishes at $t = 0$ (since the first term itself satisfies the prescribed initial conditions). For large values of t the diffusivity contribution decays like t^{-1}. However, equation (91) does not give a valid description of the diffusive decay as $t \to \infty$. This point is discussed in the following section. As is also clear from the basic equation (17), the effect of diffusivity (given by equation (91)) is proportional to the second derivative (or curvature) of the initial wave-form. In order that the second term in equation (91) be small compared with the first term, as is required for this small perturbation expansion to be valid, it is necessary that $u''(Y_M, 0)\,\delta$ be small.

The effect of large curvature in the initial wave-form is briefly discussed in § A 3 below by means of the extreme case of a wave-form with a discontinuity in slope.

A 2. *Asymptotic wave-form for large t*

The approximate wave-form described by equation (91) is obtained by taking the first terms of an expansion in powers of the (small) quantity δ at a fixed value of t. This approximate form is non-uniformly valid for large values of t. For large t, as has been shown in § 3.3, the wave-form is dominated by the conditions of uniform flow upstream and downstream of the wave, and asymptotically the (non-centred) expansion wave may be regarded as a centred expansion wave, the effect of convection being to concentrate wave curvature in the neighbourhood of the end rays. The t^{-1} decay described by equation (91) is analogous to that already found (equation (36)) for the inside of a centred wave. However, as $t \to \infty$ and the asymptotic centred wave-form is approached, diffusive regions will persist in the neighbourhood of the first and last rays, and the $t^{-\frac{1}{2}}$ decay, described in § 3 of the main text, will take place in these regions.

A 3. *Initial wave-form with discontinuity in slope*

This case has been mentioned by Lighthill (1956). He considers an initial wave-form given by

$$u(X,0) = \begin{cases} 0, & \text{for } X > 0, \\ -HX, & \text{for } X < 0, \end{cases} \tag{92}$$

where $H < 0$. The decay of u on the wave front $X = 0$ can be obtained from equation (75) and is given by (Lighthill 1956)

$$u(0,t) = \left\{\frac{2}{(1-Ht)^{\frac{1}{2}} + (1-Ht)}\right\} H \left(\frac{\delta t}{2\pi}\right)^{\frac{1}{2}}. \tag{93}$$

In this case u decays like $t^{-\frac{1}{2}}$. This is to be contrasted with the decay like t^{-1} which occurs in the case of a wave-form with a small curvature. The effect of diffusivity when the initial wave-form contains a discontinuity in slope is in fact the same as that already found (§ 3.5) at the leading and end rays of a centred expansion wave; the diffusive effect decays like $t^{-\frac{1}{2}}$ and is spread over a length increasing like $t^{\frac{1}{2}}$.

REFERENCES

Appleton, J. P. 1960 The structure of a centred rarefaction wave in an ideal dissociating gas. *Aero. Res. Counc. Rep.* no. 22095. Also *Univ. Southampton, U.S.A.A. Rep.* no. 136.

Chu, B-T. 1958 Wave propagation in a reacting mixture. *Heat Transf. Fluid Mech. Inst.* pp. 80–90.

CLARKE, J. F. 1960 The linearized flow of a dissociating gas. *J. Fluid Mech.* **7**, 577–595.

ERDELYI, A. 1956 *Asymptotic Expansions.* New York: Dover.

KANTROWITZ, A. 1958 One dimensional treatment of non-steady gas dynamics. *Fundamentals of Gas Dynamics* (ed. Emmons, H. W.). Princeton University Press.

LANDAU, L. D. & LIFSHITZ, E. M. 1959 *Fluid Mechanics.* London: Pergamon.

LIGHTHILL, M. J. 1956 Viscosity effects in sound waves of finite amplitude. *Surveys in Mechanics* (ed. Batchelor, G. K. & Davies, R. M.), p. 250. Cambridge University Press.

MOORE, F. K. & GIBSON, W. E. 1959 Propagation of weak disturbances in a gas subject to relaxation effects. *Inst. Aero. Sci. Rep.* no. 59–64.

NAPOLITANO, L. G. 1960 Non-equilibrium centred rarefaction for a reacting mixture. *Arnold Engng Dev. Center Rep.* AEDC-TN-60-129.

STULOV, V. P. 1962 The flow of an ideal dissociating gas about a convex angle corner with non-equilibrium taken into account. Unpublished R.A.E. Library Translation by B. A. Woods, from *Iz. Akad. Nauk SSSR Otdelenie Tekh. Nauk Mekh. i Mashin.* **3**, 4–10.

WHITHAM, G. B. 1959 Some comments on wave propagation and shock wave structure with applications to magnetohydrodynamics. *Comm. Pure Appl. Math.* **12**, 113–158.

WOOD, W. W. & PARKER, F. R. 1958 Structure of a centred rarefaction wave in a relaxing gas. *Phys. Fluids*, **1**, 230–241.